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# Asymptotic behavior of certain weighted quadratic and cubic variations of fractional Brownian motion

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**Abstract:** This note is devoted to a fine study of the convergence of certain weighted quadratic and cubic variations of a fractional Brownian motion  $B$  with Hurst index  $H \in (0, 1/2)$ . By means of Malliavin calculus, we show that, correctly renormalized, our weighted quadratic variations converge in  $L^2$  to an explicit limit when  $H < 1/4$ , while we show in the companion paper [14] that they converge in law when  $H > 1/4$ . Similarly, we also show that, correctly renormalized, our weighted cubic variation converge in  $L^2$  to an explicit limit when  $H < 1/6$ .

**Key words:** Fractional Brownian motion - weighted quadratic variation - weighted cubic variation - exact rate of convergence.

## 1 Introduction

The study of single path behavior of stochastic processes is often based on the study of their power variations and there exists a very extensive literature on the subject. Recall that, a real  $\kappa > 1$  being given, the  $\kappa$ -power variation of a process  $X$ , with respect to a subdivision  $\pi_n = \{0 = t_{n,0} < t_{n,1} < \dots < t_{n,n} = 1\}$  of  $[0, 1]$ , is defined to be the sum

$$\sum_{k=0}^{n-1} |X_{t_{n,k+1}} - X_{t_{n,k}}|^\kappa.$$

For simplicity, consider from now on the case where  $t_{n,k} = k/n$ , for  $n \in \mathbb{N}^*$  and  $k \in \{0, \dots, n\}$ . In this paper, we shall point out some interesting phenomena when  $X = B$  is a fractional Brownian motion with Hurst index  $H \in (0, 1/2)$  and when the value of  $\kappa$  is 2 or 3. In fact, we will also drop the absolute value (when  $\kappa = 3$ ) and we will introduce some weights. More precisely, we will consider:

$$\sum_{k=0}^{n-1} h(B_{k/n}) \Delta^\kappa B_{k/n}, \quad \kappa = 2, 3, \quad (1.1)$$

the function  $h : \mathbb{R} \rightarrow \mathbb{R}$  being assumed smooth enough and where we note, for simplicity,  $\Delta^\kappa B_{k/n}$  instead of  $(B_{(k+1)/n} - B_{k/n})^\kappa$ . Notice that, originally, the interest that we have in quantities of type (1.1) is motivated by the study of the exact rate of convergence for some approximation schemes of stochastic differential equations driven by  $B$ , see [5], [12] and [13].

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Now, let us recall some known results concerning  $\kappa$ -power variations which are today more or less classical. First, assume that the Hurst index is  $H = 1/2$ , that is  $B$  is a standard Brownian motion. Let  $\mu_\kappa$  denote the  $\kappa$ -moment of a standard Gaussian random variable  $G \sim \mathcal{N}(0, 1)$ . It is immediate, by using central limit theorem that, as  $n \rightarrow \infty$ ,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left[ n^{\kappa/2} \Delta^\kappa B_{k/n} - \mu_\kappa \right] \xrightarrow{\text{Law}} \mathcal{N}(0, \mu_{2\kappa} - \mu_\kappa^2). \quad (1.2)$$

When weights are introduced, an interesting phenomenon appears: instead of Gaussian random variables, we rather obtain mixing random variables as limit in (1.2). For instance, when  $\kappa$  is even, it is a very particular case of a more general result by Jacod [8] that we have, as  $n \rightarrow \infty$ :

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h(B_{k/n}) \left[ n^{\kappa/2} \Delta^\kappa B_{k/n} - \mu_\kappa \right] \xrightarrow{\text{Law}} \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_0^1 h(B_s) dW_s. \quad (1.3)$$

Here,  $W$  denotes another standard Brownian motion, independent of  $B$ .

Second, assume that  $H \neq 1/2$ , that is the case where the fractional Brownian motion  $B$  has not independent increments anymore. Then (1.2) has been extended by [1, 4] (see also [16] for an elegant way to obtain (1.4)-(1.5) just below) and two cases are considered according to the evenness of  $\kappa$ :

- if  $\kappa$  is even and if  $H \in (0, 3/4)$ , as  $n \rightarrow \infty$ ,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left[ n^{\kappa H} \Delta^\kappa B_{k/n} - \mu_\kappa \right] \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma_{H,\kappa}^2); \quad (1.4)$$

- if  $\kappa$  is odd and if  $H \in (0, 1/2)$ , as  $n \rightarrow \infty$ ,

$$n^{\kappa H - 1/2} \sum_{k=0}^{n-1} \Delta^\kappa B_{k/n} \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma_{H,\kappa}^2). \quad (1.5)$$

Here  $\sigma_{H,\kappa} > 0$  is a constant depending only on  $H$  and  $\kappa$ , which can be computed explicitly. In fact, one can relax the restrictive conditions made on  $H$  in (1.4)-(1.5): in this case, the normalizations are not the same anymore and, for (1.4), one obtains limits which are not Gaussian but the value at time one of an Hermite process (see [3, 19]).

Now, let us proceed with the results concerning the weighted power variations in the case where  $H \neq 1/2$ . When  $\kappa$  is even and  $H \in (1/2, 3/4)$ , then by Theorem 2 in León and Ludeña [11] we have that:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h(B_{k/n}) \left[ n^{\kappa H} \Delta^\kappa B_{k/n} - \mu_\kappa \right] \xrightarrow{\text{Law}} \sigma_{H,\kappa} \int_0^1 h(B_s) dW_s, \quad \text{as } n \rightarrow \infty, \quad (1.6)$$

where, once again,  $W$  denotes a standard Brownian motion independent of  $B$ . In other words, (1.6) shows for (1.1) a similar behavior to that observed in the standard Brownian case, compare with (1.3). See also [2] for related results on the asymptotic behavior of the  $p$ -variation of stochastic integrals with respect to  $B$ . In contradistinction, the asymptotic behavior of (1.1)

is completely different of (1.3) or (1.6) when  $H \in (0, 1/2)$  and  $\kappa$  is odd. The first result in this direction was discovered by Gradinaru, Russo and Vallois [6], when they showed that the following convergence holds when  $H = 1/4$ : as  $\varepsilon \rightarrow 0$ ,

$$\int_0^t h(B_u) \frac{(B_{u+\varepsilon} - B_u)^3}{\varepsilon} du \xrightarrow{L^2} -\frac{3}{2} \int_0^t h'(B_u) du. \quad (1.7)$$

As a continuation, Gradinaru and myself [5] improved (1.7) very recently (by working with sums instead of  $\varepsilon$ -integrals *à la* Russo-Vallois [17]). More precisely, we showed that we have, for any  $H \in (0, 1/2)$  and any odd integer  $\kappa \geq 3$ : as  $n \rightarrow \infty$ ,

$$n^{(\kappa+1)H-1} \sum_{k=0}^{n-1} h(B_{k/n}) \Delta^\kappa B_{k/n} \xrightarrow{L^2} -\frac{\mu_{\kappa+1}}{2} \int_0^1 h'(B_s) ds. \quad (1.8)$$

At this stage, we will make three comments. First, let us remark that the limits obtained in (1.7) and (1.8) do not involve an independent standard Brownian motion anymore, as it is the case in (1.3) or (1.6). Second, let us notice that (1.8) agrees with (1.5) since, when  $H \in (0, 1/2)$ , we have  $(\kappa + 1)H - 1 < \kappa H - 1/2$  and (1.8) with  $h \equiv 1$  is in fact a corollary of (1.5). Third, we want to add that exactly the same type of convergence than (1.7) had been already performed in [10], Theorem 4.1 (see also [9]), when, in (1.7), fractional Brownian motion  $B$  of Hurst index  $H = 1/4$  is replaced by an iterated Brownian motion  $Z$ . It is not very surprising, since this latter process is also centred, selfsimilar of index  $1/4$  and has stationary increments. Finally, let us also mention that Swanson announced in [18] that, in a joint work with Burdzy, he will prove that the same also holds for the solution to a stochastic heat equation.

Now, let us go back to our problem. The aim of the present work is to prove the following result:

**Theorem 1.1** *Let  $B$  be a fractional Brownian motion of Hurst index  $H$ . Then:*

1. *If  $h : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}_b^2$  and if  $H \in (0, 1/4)$ , we have, as  $n \rightarrow \infty$ :*

$$n^{2H-1} \sum_{k=0}^{n-1} h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \xrightarrow{L^2} \frac{1}{4} \int_0^1 h''(B_u) du. \quad (1.9)$$

2. *If  $h : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}_b^3$  and if  $H \in (0, 1/6)$ , we have, as  $n \rightarrow \infty$ :*

$$n^{3H-1} \sum_{k=0}^{n-1} \left[ h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] \xrightarrow{L^2} -\frac{1}{8} \int_0^1 h'''(B_u) du. \quad (1.10)$$

Before giving the proof of Theorem 1.1, let us roughly explain why (1.9) is only available when  $H < 1/4$  (of course, the same type of arguments could be also applied to understand why (1.10) is only available when  $H < 1/6$ ). For this purpose, let us first consider the case where  $B$  is a standard Brownian motion (that is the case where  $H = 1/2$ ). By using the independence of increments, we easily compute

$$E \left\{ \sum_{k=0}^{n-1} h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \right\} = 0,$$

and

$$E \left\{ \sum_{k=0}^{n-1} h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \right\}^2 = 2 E \left\{ \sum_{k=0}^{n-1} h^2(B_{k/n}) \right\} \approx 2n E \left\{ \int_0^1 h^2(B_u) du \right\}.$$

Although these two facts are of course not sufficient to guarantee that (1.3) holds when  $\kappa = 2$ , they can however roughly explain why it is true. Now, let us go back to the general case, that is the case where  $B$  is a fractional Brownian motion of index  $H \in (0, 1/2)$ . In the sequel, we will show (see Lemmas 2.2 and 2.3 for precise statements) that

$$E \left\{ \sum_{k=0}^{n-1} h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \right\} \approx \frac{1}{4} n^{-2H} \sum_{k=0}^{n-1} E [h''(B_{k/n})],$$

and, when  $H < 1/4$ :

$$\begin{aligned} E \left\{ \sum_{k=0}^{n-1} h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \right\}^2 &\approx \sum_{k \neq \ell} E \left\{ h(B_{k/n}) h(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1]^2 [n^{2H} \Delta^2 B_{\ell/n} - 1]^2 \right\} \\ &\approx \frac{1}{16} n^{-4H} \sum_{k \neq \ell} E [h''(B_{k/n}) h''(B_{\ell/n})] \\ &\approx \frac{1}{16} n^{2-4H} E \left\{ \int_{[0,1]^2} h''(B_u) h''(B_v) dudv \right\}. \end{aligned}$$

At the opposite, when  $H \in (1/4, 1/2)$ , we have (see [14] for a precise statement):

$$E \left\{ \sum_{k=0}^{n-1} h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \right\}^2 \approx n E \left\{ \sigma_H^2 \int_0^1 h^2(B_u) du \right\},$$

for a certain (explicit) constant  $\sigma_H > 0$ . Thus, the quantity  $\sum_{k=0}^{n-1} h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1]$ , when  $B$  is a fractional Brownian motion of index  $H \in (1/4, 1/2)$ , behaves as in the case where  $B$  is a standard Brownian motion, at least for the first and second order moments. In particular, it is not very surprizing to be convinced that the following convergence holds: as  $n \rightarrow \infty$ ,

$$\text{when } H \in (1/4, 1/2), \quad \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \xrightarrow{\text{Law}} \sigma_H \int_0^1 h(B_s) dW_s, \quad (1.11)$$

with  $W$  a standard Brownian motion independent of  $B$ . In fact, in the companion paper [14] (joint work with Nualart), we show that (1.11) holds.

Finally, let us remark that (1.9) agrees with (1.4), since we have  $2H - 1 < -1/2$  if and only if  $H < 1/4$  (it is an other reason which can explain the condition  $H < 1/4$  in the first point of Theorem 1.1). Thus, (1.9) with  $h \equiv 1$  is in fact a corollary of (1.4). Similarly, (1.10) agrees with (1.4), since we have  $3H - 1 < -1/2$  if and only if  $H < 1/6$  (this time, it can explain, in a sense, the condition  $H < 1/6$  in the second point of Theorem 1.1).

Now, the sequel of this note is devoted to the proof of Theorem 1.1. Instead of the *pedestrian* technique performed in [5] or [6] (as their authors called it themselves), we stress on the fact that we choosed here to use a more elegant way via Malliavin calculus. It can be viewed as an other novelty of this paper.

## 2 Proof of Theorem 1.1

### 2.1 Notations and preliminaries

We begin by briefly recalling some basic facts about stochastic calculus with respect to a fractional Brownian motion. One refers to [15] for further details. Let  $B = (B_t)_{t \in [0, T]}$  be a fractional Brownian motion with Hurst parameter  $H \in (0, 1/2)$  defined on a probability space  $(\Omega, \mathcal{A}, P)$ . We mean that  $B$  is a centered Gaussian process with the covariance function  $E(B_s B_t) = R_H(s, t)$ , where

$$R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}). \quad (2.12)$$

We denote by  $\mathcal{E}$  the set of step  $\mathbb{R}$ -valued functions on  $[0, T]$ . Let  $\mathfrak{H}$  be the Hilbert space defined as the closure of  $\mathcal{E}$  with respect to the scalar product

$$\langle \mathbf{1}_{[0, t]}, \mathbf{1}_{[0, s]} \rangle_{\mathfrak{H}} = R_H(t, s).$$

We denote by  $|\cdot|_{\mathfrak{H}}$  the associate norm. The mapping  $\mathbf{1}_{[0, t]} \mapsto B_t$  can be extended to an isometry between  $\mathfrak{H}$  and the Gaussian space  $\mathcal{H}_1(B)$  associated with  $B$ . We denote this isometry by  $\varphi \mapsto B(\varphi)$ .

Let  $\mathcal{S}$  be the set of all smooth cylindrical random variables, *i.e.* of the form

$$F = f(B(\phi_1), \dots, B(\phi_n))$$

where  $n \geq 1$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function with compact support and  $\phi_i \in \mathfrak{H}$ . The Malliavin derivative of  $F$  with respect to  $B$  is the element of  $L^2(\Omega, \mathfrak{H})$  defined by

$$D_s^B F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\phi_1), \dots, B(\phi_n)) \phi_i(s), \quad s \in [0, T].$$

In particular  $D_s^B B_t = \mathbf{1}_{[0, t]}(s)$ . As usual,  $\mathbb{D}^{1,2}$  denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 = E[F^2] + E[|D.F|_{\mathfrak{H}}^2].$$

The Malliavin derivative  $D$  verifies the chain rule: if  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $\mathcal{C}_b^1$  and if  $(F_i)_{i=1, \dots, n}$  is a sequence of elements of  $\mathbb{D}^{1,2}$  then  $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$  and we have, for any  $s \in [0, T]$ :

$$D_s \varphi(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) D_s F_i.$$

The divergence operator  $I$  is the adjoint of the derivative operator  $D$ . If a random variable  $u \in L^2(\Omega, \mathfrak{H})$  belongs to the domain of the divergence operator, that is if it verifies

$$|E\langle DF, u \rangle_{\mathfrak{H}}| \leq c_u \|F\|_{L^2} \quad \text{for any } F \in \mathcal{S},$$

then  $I(u)$  is defined by the duality relationship

$$E(FI(u)) = E\langle DF, u \rangle_{\mathfrak{H}},$$

for every  $F \in \mathbb{D}^{1,2}$ .

## 2.2 Proof of (1.9)

From now on, we assume that  $H \in (0, \frac{1}{4})$ . For simplicity, we note  $\delta_{k/n} = \mathbf{1}_{[k/n, k+1/n]}$  and  $\varepsilon_{k/n} = \mathbf{1}_{[0, k/n]}$ , and we note  $\sum_k$  instead of  $\sum_{k=0}^{n-1}$ ,  $\sum_{k < \ell}$  instead of  $\sum_{0 \leq k < \ell \leq n-1}$  and  $\sum_{\ell \neq k}$  instead of  $\sum_{0 \leq k < \ell \leq n-1} + \sum_{0 \leq \ell < k \leq n-1}$ . Also  $C$  will denote a generic constant that can be different from line to line. We will need several lemmas.

**Lemma 2.1** *For  $x \geq 0$ , we have:*

$$|(x+1)^{2H} - x^{2H}| \leq 1$$

while, for  $x \geq 1$ , we have

$$|(x+1)^{2H} + (x-1)^{2H} - 2x^{2H}| \leq (2 - 2^{2H}).$$

**Proof.** For  $x \geq 0$ , we can write:

$$|(x+1)^{2H} - x^{2H}| = 2H \int_0^1 \frac{du}{(x+u)^{1-2H}} \leq 2H \int_0^1 \frac{du}{u^{1-2H}} = 1.$$

Similarly, for  $x \geq 1$ :

$$\begin{aligned} |(x+1)^{2H} + (x-1)^{2H} - 2x^{2H}| &= 2H|2H-1| \int_{[0,1]^2} \frac{dudv}{(x+u-v)^{2-2H}} \\ &\leq 2H|2H-1| \int_{[0,1]^2} \frac{dudv}{(1+u-v)^{2-2H}} = 2 - 2^{2H}. \end{aligned}$$

□

**Lemma 2.2** *For  $h, g : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}_b^2$ , we have*

$$\sum_{\ell \neq k} E \{ h(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \} = \frac{1}{4} n^{-2H} \sum_{\ell \neq k} E \{ h''(B_{k/n}) g(B_{\ell/n}) \} + o(n^{2-2H}), \quad (2.13)$$

$$\sum_k E \{ h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \} = \frac{1}{4} n^{-2H} \sum_k E \{ h''(B_{k/n}) \} + o(n^{1-2H}), \quad (2.14)$$

**Proof.** Let us first prove (2.13). For  $0 \leq \ell, k \leq n-1$ , we can write:

$$\begin{aligned} &E \{ h(B_{k/n}) g(B_{\ell/n}) n^{2H} \Delta^2 B_{k/n} \} \\ &= E \{ h(B_{k/n}) g(B_{\ell/n}) n^{2H} \Delta B_{k/n} I(\delta_{k/n}) \} \\ &= E \{ h'(B_{k/n}) g(B_{\ell/n}) n^{2H} \Delta B_{k/n} \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} + E \{ h(B_{k/n}) g'(B_{\ell/n}) n^{2H} \Delta B_{k/n} \} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &+ E \{ h(B_{k/n}) g(B_{\ell/n}) \}. \end{aligned}$$

Thus,

$$\begin{aligned}
& n^{-2H} E \{ h(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \} \\
&= E \{ h'(B_{k/n}) g(B_{\ell/n}) I(\delta_{k/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} + E \{ h(B_{k/n}) g'(B_{\ell/n}) I(\delta_{k/n}) \} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&= E \{ h''(B_{k/n}) g(B_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 + 2 E \{ h'(B_{k/n}) g'(B_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&+ E \{ h(B_{k/n}) g''(B_{\ell/n}) \} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2.
\end{aligned}$$

But

$$\begin{aligned}
\langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} &= \frac{1}{2} n^{-2H} ((k+1)^{2H} - k^{2H} - 1) \\
\langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} &= \frac{1}{2} n^{-2H} ((k+1)^{2H} - k^{2H} - |\ell - k - 1|^{2H} + |\ell - k|^{2H}).
\end{aligned}$$

In particular,

$$\begin{aligned}
& \left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 - \frac{1}{4} n^{-4H} \right| \\
&= \left| \frac{1}{4} n^{-4H} \left( ((k+1)^{2H} - k^{2H})^2 - 2((k+1)^{2H} - k^{2H}) \right) \right| \\
&\leq \frac{3}{4} n^{-4H} ((k+1)^{2H} - k^{2H}), \quad \text{by Lemma 2.1,}
\end{aligned}$$

and, consequently:

$$n^{2H} \sum_{k \neq \ell} \left| E \{ h''(B_{k/n}) g(B_{\ell/n}) \} (\langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 - \frac{1}{4} n^{-4H}) \right| \leq C n^{1-2H} \sum_{k=0}^{n-1} ((k+1)^{2H} - k^{2H}) = Cn.$$

Similarly, using again Lemma 2.1, we deduce:

$$\begin{aligned}
& \left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| + \left| \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right| \\
&\leq C n^{-4H} (|(k+1)^{2H} - k^{2H}| + ||\ell - k|^{2H} - |\ell - k - 1|^{2H}|).
\end{aligned}$$

Since

$$\sum_{k \neq \ell} ||\ell - k|^{2H} - |\ell - k - 1|^{2H}| = 2 \sum_{k < \ell} ((\ell - k)^{2H} - (\ell - k - 1)^{2H}) = 2 \sum_{\ell=1}^{n-1} \ell^{2H} \leq 2n^{2H+1},$$

we have that

$$n^{2H} \sum_{k \neq \ell} (|2E \{ h'(B_{k/n}) g'(B_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}| + |E \{ h(B_{k/n}) g''(B_{\ell/n}) \} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2|) \leq Cn.$$

In particular, equality (2.13) follows, since  $n = o(n^{2-2H})$ . The proof of (2.14), corresponding to the case where  $g \equiv 1$ , is simpler and similar.  $\square$



**Lemma 2.3** For  $h, g : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}_b^4$ , we have

$$\begin{aligned} & \sum_{\ell \neq k} E \{ h(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] [n^{2H} \Delta^2 B_{\ell/n} - 1] \} \\ &= \frac{1}{16} n^{-4H} \sum_{\ell \neq k} E \{ h''(B_{k/n}) g''(B_{\ell/n}) \} + o(n^{2-4H}) \end{aligned} \quad (2.15)$$

**Proof.** For  $0 \leq \ell, k \leq n-1$ , we can write:

$$\begin{aligned} & E \{ h(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] n^{2H} \Delta^2 B_{\ell/n} \} \\ &= E \{ h(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] n^{2H} \Delta B_{\ell/n} I(\delta_{\ell/n}) \} \\ &= E \{ h'(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] n^{2H} \Delta B_{\ell/n} \} \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\ &+ E \{ h(B_{k/n}) g'(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] n^{2H} \Delta B_{\ell/n} \} \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\ &+ 2 E \{ h(B_{k/n}) g(B_{\ell/n}) n^{4H} \Delta B_{k/n} \Delta B_{\ell/n} \} \langle \delta_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} + E \{ h(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \}. \end{aligned}$$

Thus,

$$\begin{aligned} & E \{ h(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] [n^{2H} \Delta^2 B_{\ell/n} - 1] \} \\ &= E \{ h'(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] n^{2H} I(\delta_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\ &+ E \{ h(B_{k/n}) g'(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] n^{2H} I(\delta_{\ell/n}) \} \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\ &+ 2 E \{ h(B_{k/n}) g(B_{\ell/n}) n^{4H} \Delta B_{k/n} I(\delta_{\ell/n}) \} \langle \delta_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\ &= n^{2H} E \{ h''(B_{k/n}) g(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \} \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 \\ &+ 2n^{2H} E \{ h'(B_{k/n}) g'(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \} \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\ &+ 4n^{4H} E \{ h'(B_{k/n}) g(B_{\ell/n}) \Delta B_{k/n} \} \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \delta_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\ &+ 4n^{4H} E \{ h(B_{k/n}) g'(B_{\ell/n}) \Delta B_{k/n} \} \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \delta_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\ &+ 2n^{4H} E \{ h(B_{k/n}) g(B_{\ell/n}) \} \langle \delta_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 \\ &+ n^{2H} E \{ h(B_{k/n}) g''(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \} \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 \triangleq \sum_{i=1}^6 R_{k,\ell,n}^i. \end{aligned}$$

We claim that, for any  $1 \leq i \leq 5$ , we have  $\sum_{k \neq \ell} |R_{k,\ell,n}^i| = o(n^{2-4H})$ . Let us consider, for instance, the case where  $i = 1$ . We have, using Lemma 2.1:

$$|R_{k,\ell,n}^1| \leq C |\langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}|.$$

As in the proof of Lemma 2.2, we deduce that  $\sum_{k \neq \ell} |R_{k,\ell,n}^1| \leq Cn = o(n^{2-4H})$ , since  $H < \frac{1}{4}$ . It remains to consider  $R_{k,\ell,n}^6$ . By (the proof of) Lemma 2.2, we can write

$$\sum_{k \neq \ell} R_{k,\ell,n}^6 = n^{4H} \sum_{k \neq \ell} E \{ h''(B_{k/n}) g''(B_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 + o(n^{2-4H}).$$

But, from Lemma 2.1 and by developing, we deduce

$$\left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 - \frac{1}{16} n^{-8H} \right| \leq C n^{-8H} ((k+1)^{2H} - k^{2H} + (\ell+1)^{2H} - \ell^{2H}).$$

This yields

$$\sum_{k \neq \ell} R_{k,\ell,n}^6 = \frac{1}{16} n^{-4H} \sum_{k \neq \ell} E \{ h''(B_{k/n}) g''(B_{\ell/n}) \} + o(n^{2-4H}).$$

The proof of Lemma 2.3 is done.  $\square$

We are now in position to prove (1.9). Using Lemma 2.3, we have on one hand:

$$\begin{aligned} & E \left\{ n^{2H-1} \sum_k h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \right\}^2 \\ &= n^{4H-2} \sum_k E \left\{ h^2(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1]^2 \right\} \\ &+ n^{4H-2} \sum_{\ell \neq k} E \{ h(B_{k/n}) h(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] [n^{2H} \Delta^2 B_{\ell/n} - 1] \} \\ &= \frac{1}{16} n^{-2} \sum_{\ell \neq k} E \{ h''(B_{k/n}) h''(B_{\ell/n}) \} + O(n^{4H-1}). \end{aligned} \tag{2.16}$$

Using Lemma 2.2, we have on the other hand:

$$\begin{aligned} & E \left\{ n^{2H-1} \sum_k h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \times \frac{1}{4n} \sum_{\ell} h''(B_{\ell/n}) \right\} \\ &= \frac{n^{2H-2}}{4} \left( \sum_k E \{ (h h'')(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \} + \sum_{k \neq \ell} E \{ h(B_{k/n}) h''(B_{\ell/n}) [n^{2H} \Delta^2 B_{k/n} - 1] \} \right) \\ &= \frac{1}{16} n^{-2} \sum_{k \neq \ell} E \{ h''(B_{k/n}) h''(B_{\ell/n}) \} + o(1). \end{aligned} \tag{2.17}$$

Now, we easily deduce (1.9). Indeed, thanks to (2.16)-(2.17), we obtain, by developing the square and by remembering that  $H < 1/4$ , that

$$E \left\{ n^{2H-1} \sum_k h(B_{k/n}) [n^{2H} \Delta^2 B_{k/n} - 1] - \frac{1}{4n} \sum_k h''(B_{k/n}) \right\}^2 \longrightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Since  $\frac{1}{4n} \sum_k h''(B_{k/n}) \xrightarrow{L^2} \frac{1}{4} \int_0^1 h''(B_u) du$  as  $n \rightarrow \infty$ , we have finally proved that (1.9) holds.

## 2.3 Proof of (1.10)

As in the previous section, we first need two technical lemmas.

**Lemma 2.4** For  $h, g : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}_b^3$ , we have

$$\begin{aligned} & \sum_{\ell \neq k} E \{ h(B_{k/n}) g(B_{\ell/n}) n^{3H} \Delta^3 B_{k/n} \} \\ &= -\frac{3}{2} n^{-H} \sum_{\ell \neq k} E \{ h'(B_{k/n}) g(B_{\ell/n}) \} - \frac{1}{8} n^{-3H} \sum_{\ell \neq k} E \{ h'''(B_{k/n}) g(B_{\ell/n}) \} + o(n^{2-3H}). \end{aligned} \tag{2.18}$$

and

$$\sum_{\ell \neq k} E \{ h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} \} = -\frac{3}{2} n^{-H} \sum_k E \{ h'(B_{k/n}) \} - \frac{1}{8} n^{-3H} \sum_k E \{ h'''(B_{k/n}) \} + o(n^{1-3H}).$$

**Proof.** For  $0 \leq \ell, k \leq n-1$ , we can write:

$$\begin{aligned} & E \{ h(B_{k/n}) g(B_{\ell/n}) n^{3H} \Delta^3 B_{k/n} \} \\ &= E \{ h(B_{k/n}) g(B_{\ell/n}) n^{3H} \Delta^2 B_{k/n} I(\delta_{k/n}) \} \\ &= E \{ h'(B_{k/n}) g(B_{\ell/n}) n^{3H} \Delta^2 B_{k/n} \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} + E \{ h(B_{k/n}) g'(B_{\ell/n}) n^{3H} \Delta^2 B_{k/n} \} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &+ 2 E \{ h(B_{k/n}) g(B_{\ell/n}) n^H \Delta B_{k/n} \} \\ &= E \{ h''(B_{k/n}) g(B_{\ell/n}) n^{3H} \Delta B_{k/n} \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 + 2 E \{ h'(B_{k/n}) g'(B_{\ell/n}) n^{3H} \Delta B_{k/n} \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &+ 3 n^H E \{ h'(B_{k/n}) g(B_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} + E \{ h(B_{k/n}) g''(B_{\ell/n}) n^{3H} \Delta B_{k/n} \} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\ &+ 3 n^H E \{ h(B_{k/n}) g'(B_{\ell/n}) \} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &= n^{3H} E \{ h'''(B_{k/n}) g(B_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^3 + 3 n^{3H} E \{ h''(B_{k/n}) g'(B_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &+ 3 n^{3H} E \{ h'(B_{k/n}) g''(B_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 + 3 n^H E \{ h'(B_{k/n}) g(B_{\ell/n}) \} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &+ n^{3H} E \{ h(B_{k/n}) g'''(B_{\ell/n}) \} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^3 + 3 n^H E \{ h(B_{k/n}) g'(B_{\ell/n}) \} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}. \end{aligned}$$

We can now finish as in the proof of Lemma 2.2.  $\square$

Similarly, we can prove:

**Lemma 2.5** For  $h, g : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}_b^3$ , we have

$$\begin{aligned} & \sum_{\ell \neq k} E \{ h(B_{k/n}) g(B_{\ell/n}) n^{3H} \Delta^3 B_{k/n} n^{3H} \Delta^3 B_{\ell/n} \} \\ &= \frac{9}{4} n^{-2H} \sum_{\ell \neq k} E \{ h'(B_{k/n}) g'(B_{\ell/n}) \} + \frac{3}{16} n^{-4H} \sum_{\ell \neq k} E \{ h'(B_{k/n}) g'''(B_{\ell/n}) \} \\ &+ \frac{3}{16} n^{-4H} \sum_{\ell \neq k} E \{ h'''(B_{k/n}) g'(B_{\ell/n}) \} + \frac{1}{64} n^{-6H} \sum_{\ell \neq k} E \{ h'''(B_{k/n}) g'''(B_{\ell/n}) \} + o(n^{2-6H}). \end{aligned} \quad (2.19)$$

**Proof.** Left to the reader: use the same technic than in the proof of Lemma 2.4.  $\square$

We are now in position to prove (1.10). Using Lemmas 2.4 and 2.5, we have on one hand

$$\begin{aligned} & E \left\{ n^{3H-1} \sum_k \left[ h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] \right\}^2 \\ &= n^{6H-2} \sum_k E \left[ h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right]^2 \\ &+ n^{6H-2} \sum_{\ell \neq k} E \left[ h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] \left[ h(B_{\ell/n}) n^{3H} \Delta^3 B_{\ell/n} + \frac{3}{2} h'(B_{\ell/n}) n^{-H} \right] \\ &= \frac{1}{64} n^{-2} \sum_{\ell \neq k} E \{ h'''(B_{k/n}) h'''(B_{\ell/n}) \} + O(n^{6H-1}). \end{aligned} \quad (2.20)$$

On the other hand, we have:

$$\begin{aligned}
& E \left\{ n^{3H-1} \sum_k \left[ h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] \times \frac{-1}{8n} \sum_\ell h'''(B_{\ell/n}) \right\} \quad (2.21) \\
&= -\frac{n^{3H-2}}{8} \left( \sum_k E \left[ (h h''')(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} (h' h''')(B_{k/n}) n^{-H} \right] \right. \\
&\quad \left. + \sum_{k \neq \ell} E \left[ h(B_{k/n}) h'''(B_{\ell/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) h'''(B_{\ell/n}) n^{-H} \right] \right) \\
&= \frac{1}{64} n^{-2} \sum_{k \neq \ell} E \{ h'''(B_{k/n}) h'''(B_{\ell/n}) \} + o(1).
\end{aligned}$$

Now, we easily deduce (1.10). Indeed, thanks to (2.20)-(2.21), we obtain, by developing the square and by remembering that  $H < 1/6$ , that

$$E \left\{ n^{3H-1} \sum_k \left[ h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] + \frac{1}{8n} \sum_k h'''(B_{k/n}) \right\}^2 \longrightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Since  $-\frac{1}{8n} \sum_k h'''(B_{k/n}) \xrightarrow{L^2} -\frac{1}{8} \int_0^1 h'''(B_u) du$  as  $n \rightarrow \infty$ , we have finally proved that (1.10) holds.

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